

Families of 0-dimensional subspaces on supercurves of dimension $(1|1)$

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Abstract

In the present paper we prove that the Hilbert scheme of 0-dimensional subspaces on supercurves of dimension $(1|1)$ exists and it is smooth. We show that the Hilbert scheme is not split in general.

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1 Introduction

Supergeometry is a \mathbb{Z}_2 -graded generalization of the ordinary geometry. For references, see [1, 2, 3]. After it has been shown that the supermoduli space is not projected [4], the importance of establishing mathematical foundations about supermoduli spaces, (analytic) superspaces, supermanifolds, etc. has increased.

The construction for the Hilbert scheme of ordinary projective space was developed by Alexander Grothendieck [5]. In this paper, we first show the existence of the (analytic) Hilbert scheme $\text{Hilb}(S)$ of 0-dimensional subspaces on a supercurve S of dimension $(1|1)$ (see 2.3 for definition). This Hilbert scheme can be broken up into disjoint union $\text{Hilb}(S) = \bigcup_{(p,q)} \text{Hilb}^{p|q}(S)$ where $\text{Hilb}^{p|q}(S)$ is a smooth superspace of $\dim (p|p)$. This can be seen as an analogous result to the ordinary case that the Hilbert scheme of p points on a smooth surface is smooth and has dimension $2p$ [6].

Furthermore, we use the defining equation of the Hilbert scheme to see if $\text{Hilb}^{p|q}(S)$ is split or not. We show that, for any k , the Hilbert scheme $\text{Hilb}^{1|1}\Pi(\mathcal{O}_{\mathbb{P}^1}(k))$ is split, whereas the Hilbert scheme $\text{Hilb}^{2|1}\Pi(\mathcal{O}_{\mathbb{P}^1}(k))$ is not split. In fact, $\text{Hilb}^{2|1}\Pi(\mathcal{O}_{\mathbb{P}^1}(k))$ is not even projected. This also guarantees that any superspace containing $\text{Hilb}^{2|1}\Pi(\mathcal{O}_{\mathbb{P}^1}(k))$ is not split.

2 Backgrounds

2.1 Supergeometry

Ordinary geometry can be generalized by supergeometry which has an additional anti-commutative part. Definitions about supergeometry can be found, for example, in Manin's book [1]. In this section, we will review definitions of major terms.

Definition 2.1. A *superspace* is a pair (S, \mathcal{O}_S) where S is a topological space and $\mathcal{O}_S = \mathcal{O}_{S,0} \oplus \mathcal{O}_{S,1}$ is a sheaf of supercommutative rings which is a locally ringed space. Let \mathcal{J} be the ideal generated by the odd part $\mathcal{O}_{S,1}$. The bosonic space $S_b \subset S$ is the closed subspace $(S, \mathcal{O}_S/\mathcal{J})$.

From now on, we will only consider the superspaces over \mathbb{C} .

Similar to the ordinary space, locally free sheaves on superspaces can be defined. The only difference is that they have even and odd ranks. For example, a free sheaf of rank $(p|q)$ on a superspace S is $\mathcal{O}_S^p \oplus \Pi \mathcal{O}_S^q$, where $\Pi \mathcal{O}_S^q$ is the parity reversed bundle of \mathcal{O}_S^q .

A superspace (S, \mathcal{O}_S) is said to be *split* if it is isomorphic to $S(S_b, \mathcal{E}) := (S_b, \wedge^\bullet \mathcal{E}^\vee)$, where \mathcal{E} is a locally free sheaf of \mathcal{O}_{S_b} -modules. Let m be the *dimension* of S_b and let n be the rank of \mathcal{E} . Then the dimension of the superspace (S, \mathcal{O}_S) is $(m|n)$. We say a superspace (S, \mathcal{O}_S) is *locally split* if for any $x \in S$ there is a neighborhood U of x such that $(U, \mathcal{O}_S|_U)$ is split.

For the rest of this paper, we mainly discuss about analytic superspaces. One basic property of analytic superspace is that, like ordinary analytic spaces, we can take local coordinates.

Example 2.1. An analytic affine superspace

$$\mathbb{C}^{m|n} = (\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^{m|n}}) = S(\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^m}^n)$$

is one of the simplest examples of the split superspace. Here, $\mathcal{O}_{\mathbb{C}^m}$ represents the sheaf of analytic functions and the structure sheaf is given by $\mathcal{O}_{\mathbb{C}^{m|n}} = \mathcal{O}_{\mathbb{C}^m}[\theta_1, \dots, \theta_n]$.

Let U be an open subset of \mathbb{C}^m . For an ideal $I \subset \mathcal{O}_{\mathbb{C}^{m|n}}(U)$, we can define an closed subset $Z(I) := Z(I \cap \mathcal{O}_{\mathbb{C}^m}(U)) \subset \mathbb{C}^m$. The analytic subspace defined by I on U is the superspace $(Z(I), \mathcal{O}_Z := \mathcal{O}_U/I)$.

Definition 2.2. An analytic superspace (S, \mathcal{O}_S) is a superspace which is locally isomorphic to some analytic subspace.

We say that an analytic superspace (S, \mathcal{O}_S) is *smooth* at $x \in S$ if there is an open neighborhood U of x such that $(U, \mathcal{O}_S|_U)$ is isomorphic to an open subspace of some analytic affine superspace. An analytic superspace (S, \mathcal{O}_S) is called smooth if it is smooth at every point in S .

A locally split analytic superspace (S, \mathcal{O}_S) is called a *supermanifold* if S_b is a manifold. Note that a locally split analytic superspace (S, \mathcal{O}_S) is smooth if and only if it is a supermanifold.

Definition 2.3. A supercurve is a complex supermanifold of dimension $(1|n)$ for some non-negative integer n .

We will focus on analytic superspaces and will drop “analytic” for simplicity, and denoting it as superspaces.

2.2 Hilbert Scheme

Definition 2.4.

- i) Let S be a superspace. The *Hilbert functor* $\mathcal{H}_S^{p|q}$ is the contravariant functor from the category \mathfrak{S} of superspaces to the category of sets defined as follows:

$$\mathcal{H}_S^{p|q}(B) = \left\{ \begin{array}{c} \mathcal{Z} \hookrightarrow S \times B \\ \downarrow \pi \quad \swarrow \\ B \end{array} \middle| \begin{array}{l} \mathcal{Z} \text{ is a closed subspace of} \\ S \times B \text{ and } \pi_* \mathcal{O}_{\mathcal{Z}} \text{ is a locally} \\ \text{free } \mathcal{O}_B\text{-module of rank } (p|q) \end{array} \right\}$$

The morphism is defined by the pullback

$$\mathcal{H}_S^{p|q}(f) = f^* : \mathcal{H}_S^{p|q}(B) \rightarrow \mathcal{H}_S^{p|q}(C)$$

where $f : C \rightarrow B$ and $B, C \in \mathfrak{S}$.

- ii) Suppose that the Hilbert functor $\mathcal{H}_S^{p|q}$ is representable by the superspace $\text{Hilb}^{p|q}(S)$. We call this the *analytic Hilbert scheme*, abbreviated to the Hilbert scheme.

Example 2.2. The Hilbert functor $\mathcal{H}_{\mathbb{C}^{1|1}}^{1|1}$ is representable by $\mathbb{C}^{1|1}$.

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & \mathbb{C}_{x|\theta}^{1|1} \times \mathbb{C}_{a|\alpha}^{1|1} \\ \downarrow \pi & & \swarrow \\ \mathbb{C}_{a|\alpha}^{1|1} & & \end{array}$$

where the subscripts define coordinates and \mathcal{Z} is defined by the ideal $(x + a + \alpha\theta)$. This can be checked directly, or as a consequence of the proof of Theorem 2.5.

We prove the following theorem.

Theorem 2.5. *Let S be a supercurve. Then the functor $\mathcal{H}_S^{p|q}$ is representable by the smooth superspace $\text{Hilb}^{p|q}(S)$ of dimension $(p|p)$.*

2.3 Obstruction class for splitting

In this section, we review the definition an obstruction class which has a critical role in verifying splitness of supermanifolds [4].

Consider a supermanifold $S = (M, \mathcal{O}_S)$ and let $\mathcal{J} \subset \mathcal{O}_S$ be the sheaf of ideals generated by all nilpotents. Observe that S is locally isomorphic to the split model $S(M, \mathcal{E})$, where \mathcal{E} is defined by $\mathcal{E} = (\mathcal{J}/\mathcal{J}^2)^\vee$. As shown in [4], it induces an element $\phi \in H^1(M, \text{Aut}(\wedge^\bullet \mathcal{E}))$. Let G be the set of automorphisms of $\wedge^\bullet \mathcal{E}$ which act trivially on M and \mathcal{E} . Since the induced automorphism preserves M and \mathcal{E} , we can say that $\phi \in H^1(M, G)$. Conversely, an element S in $H^1(M, G)$, with the ideal \mathcal{J} generated by all nilpotents, gives a superspace which is locally isomorphic to $S(M, \mathcal{E})$ and $\mathcal{J}/\mathcal{J}^2 \simeq \mathcal{E}^\vee$.

Consider the filtration of S

$$M = S^{(0)} \subset S^{(1)} \subset \dots \subset S^{(n)} = S$$

where $S^{(i)} = (M, \mathcal{O}_S/\mathcal{J}^{i+1})$ and $n = \text{rank } \mathcal{E}$. Define $G^{(i)}$ to be the set of automorphisms of S which are trivial on $S^{(i-1)}$ for $i = 2, 3, \dots$. Note that there is an isomorphism

$$G^{(i)}/G^{(i+1)} \simeq T_{(-)i}M \otimes \wedge^i \mathcal{E}$$

where $T_{(-)i} = T_-$ is an odd tangent space if i is odd and $T_{(-)i} = T_+$ is an even tangent space if i is even. Moreover, this isomorphism induces an exact sequence

$$H^1(M, G^{(i+1)}) \rightarrow H^1(M, G^{(i)}) \xrightarrow{\omega} H^1(M, T_{(-)i}M \otimes \wedge^i \mathcal{E})$$

Start with $\psi^{(1)} := \phi$ and we define obstruction classes inductively. Suppose we have $\phi^{(i-1)} \in H^1(M, G^{(i)})$. If $\omega(\phi^{(i-1)}) = 0$, then there exists $\phi^{(i)} \in H^1(M, G^{(i+1)})$ such that $\phi^{(i)}$ maps to $\phi^{(i-1)}$.

The i -th obstruction class is defined by

$$\omega_i := \omega(\phi^{(i-1)}) \in T_{(-)^i} M \otimes \wedge^i \mathcal{E}$$

Observe $G^{(2)} = G$ and $\phi^{(1)} = \phi$.

In section 5.1, we will use the fact that if the second obstruction class ω_2 is not vanishing, then the superspace is not split.

3 Local structure of the Hilbert schemes

The Hilbert scheme of the affine space $\mathbb{C}^{1|1}$ is the basis for the construction of the 0-dimensional family on supercurves. Let $(x|\theta)$ be coordinates on $\mathbb{C}^{1|1}$.

Lemma 3.1. *Let $\mathcal{Y} \subset \mathbb{C}^{1|1}$ be a subspace such that $\dim_{\mathbb{C}} H^0(\mathbb{C}^{1|1}, \mathcal{O}_{\mathcal{Y}}) = (p|q)$. Then $H^0(\mathbb{C}^{1|1}, \mathcal{O}_{\mathcal{Y}})$ has basis $1, x, \dots, x^{p-1}, \theta, \theta x, \dots, \theta x^{q-1}$ as a \mathbb{C} -vector space.*

Lemma 3.2. *Let $X = (x_{ij})$ be an $n \times n$ (left) invertible matrix and let $\Gamma = (\gamma_{ij})$ be an $n \times n$ matrix such that $\gamma_{ij}^2 = 0$ for each i and j , then $X + \Gamma$ is (left) invertible.*

Proposition 3.3. *Pick $[\mathcal{Z} \xrightarrow{\pi} B] \in \mathcal{H}_{\mathbb{C}^{1|1}}^{p|q}(B)$, then $\pi_* \mathcal{O}_{\mathcal{Z}}$ is a free \mathcal{O}_B -module generated by $1, x, \dots, x^{p-1}, \theta, \theta x, \dots, \theta x^{q-1}$, i.e., $\pi_* \mathcal{O}_{\mathcal{Z}}$ is isomorphic to $\mathcal{O}_B^p \oplus \Pi \mathcal{O}_B^q$.*

Proof. Observe that the stalk $(\pi_* \mathcal{O}_{\mathcal{Z}})_t$ is a free $\mathcal{O}_{B,t}$ -module of rank $(p|q)$ for each $t \in B$. Let $M_{n,m}(R) = (a_{ij})$ denote an $n \times m$ matrix, where $a_{ij} \in R$. Let $\{f_i \in (\pi_* \mathcal{O}_{\mathcal{Z}})_t^0\}_{i=1}^p$ be even generators and let $\{g_j \in (\pi_* \mathcal{O}_{\mathcal{Z}})_t^1\}_{j=1}^q$ be odd generators. Denote $(f_i)_{i=1}^p$, $(g_j)_{j=1}^q$, $(x^i)_{i=0}^{p-1}$ and $(x^j \theta)_{j=0}^{q-1}$ by F , G , X and $X\Theta$. Then we can find $A \in M_{p,p}((\mathcal{O}_{B,t})^0)$, $B \in M_{p,q}((\mathcal{O}_{B,t})^1)$, $C \in M_{q,p}((\mathcal{O}_{B,t})^1)$ and $D \in M_{q,q}((\mathcal{O}_{B,t})^0)$ such that

$$\begin{pmatrix} X \\ X\Theta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} F \\ G \end{pmatrix}$$

Consider the surjection to the fiber \mathcal{Z}_t at t

$$\mathcal{O}_{\mathcal{Z}} \rightarrow \mathcal{O}_{\mathcal{Z}_t} \rightarrow 0$$

Then this map induces the diagram

$$\begin{array}{ccc} (\pi_* \mathcal{O}_{\mathcal{Z}})_t & \xrightarrow{\phi} & (\pi_* \mathcal{O}_{\mathcal{Z}_t})_t \\ q_1 \downarrow & & \downarrow q_2 \\ \frac{(\pi_* \mathcal{O}_{\mathcal{Z}})_t}{\mathfrak{m}_t(\pi_* \mathcal{O}_{\mathcal{Z}})_t} & \xrightarrow{\tilde{\phi}} & \frac{(\pi_* \mathcal{O}_{\mathcal{Z}_t})_t}{\mathfrak{m}_t(\pi_* \mathcal{O}_{\mathcal{Z}_t})_t} \end{array}$$

where \mathfrak{m}_t is the maximal ideal of the local ring $\mathcal{O}_{B,t}$. Observe that $\tilde{\phi}$ is a \mathbb{C} -linear isomorphism and, by the lemma 3.1, $\frac{(\pi_* \mathcal{O}_{\mathcal{Z}_t})_t}{\mathfrak{m}_t(\pi_* \mathcal{O}_{\mathcal{Z}_t})_t}$ is generated by $1, x, \dots, x^{p-1}$ and $\theta, \theta x, \dots, \theta x^{q-1}$.

Let \bar{h} represent the image of h by the quotient map q_k and let $A = (a_{ij})$ and $D = (d_{ij})$. Then we have

$$\bar{A}F = X \text{ and } \bar{D}G = X\Theta$$

where $\bar{A} = (\bar{a}_{ij})$ and $\bar{D} = (\bar{d}_{ij})$ are invertible. By the lemma 3.2, A , D and $-CA^{-1}B + D$ are invertible. Therefore, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has the left inverse $\begin{pmatrix} A^{-1} + A^{-1}B(-CA^{-1}B + D)^{-1}CA^{-1} & -A^{-1}B(-CA^{-1}B + D)^{-1} \\ -(-CA^{-1}B + D)^{-1}CA^{-1} & (-CA^{-1}B + D)^{-1} \end{pmatrix}$

□

3.1 Flattening Stratification

Lemma 3.4. (Nakayama's lemma [7]) *Let R be any ring R with the Jacobson ideal $J(R) \subset R$. For any finitely generated left R -module M , $J(R)M = M$ implies $M = 0$.*

Flattening stratification for superspaces can be done in a similar way to the ordinary cases. ([10])

Theorem 3.5. (Flattening Stratification) *Let \mathcal{B} be a Noetherian superspace and \mathcal{F} be a coherent sheaf of modules on $\mathbb{C}^{1|1} \times \mathcal{B}$. Suppose that the support of each fiber of the projection map $\pi : \mathcal{F} \rightarrow \mathcal{B}$ is zero dimensional. For each $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, there is a locally closed subspace $\mathcal{B}_{(p,q)} \subset \mathcal{B}$ such that*

i) $\pi_* \mathcal{F}|_{\mathcal{B}_{(p,q)}}$ is locally free of rank $(p|q)$,

ii) $\bigcup_{p,q} \mathcal{B}_{(p,q)} = \mathcal{B}$

iii) Such stratification is universal. (I.e. for any morphism $f : C \rightarrow B$, the induced map $f^*\mathcal{F} \rightarrow C$ is flat of rank $(p|q)$ if and only if f factors through $C \rightarrow B_{(p,q)} \hookrightarrow B$)

Proof. Pick any $b \in \mathcal{B}$ such that $\dim_{k(b)} \mathcal{F}_b \times_{\mathcal{O}_B} \text{Spec } k(b) = (p|q)$, where $k(b)$ is the quotient field at b . By the lemma 3.4, we can find some neighborhood U of b and the exact sequence

$$\mathcal{O}_U^s \oplus \Pi \mathcal{O}_U^t \xrightarrow{\sigma} \mathcal{O}_U^p \oplus \Pi \mathcal{O}_U^q \xrightarrow{\zeta} \mathcal{F}|_U \rightarrow 0$$

For any morphism $f : V \rightarrow U$ to the subspace $U = (U, \mathcal{O}_B|_U)$, we get the induced exact sequence

$$\mathcal{O}_V^s \oplus \Pi \mathcal{O}_V^t \xrightarrow{f^*\sigma} \mathcal{O}_V^p \oplus \Pi \mathcal{O}_V^q \xrightarrow{f^*\zeta} f^*\mathcal{F} \rightarrow 0$$

Note that $f^*\mathcal{F}$ is free of rank $(p|q)$ if and only if $f^*\sigma = 0$. Let σ be represented by the matrix (σ_{ij}) . If $f^*\sigma_{ij} = 0$ for all i and j , then f factors through the inclusion $U_\sigma \hookrightarrow U$ where U_σ is the closed subspace of U defined by the ideal $I_\sigma = (\sigma_{ij})$, and vice versa. Therefore, $f^*\mathcal{F}$ is flat over V if and only if f factors through $U_\sigma \hookrightarrow U$. It proves that U_σ represents the functor \mathcal{G}_U where $\mathcal{G}_U(f : V \rightarrow U) = \{f^*\mathcal{F} \rightarrow V \text{ is flat of rank } (p|q)\}$, i.e. U_σ is universal. Moreover, the universality guarantees that we can glue all U_σ 's with fixed $(p|q)$ and $B_{p,q} := \cup_\sigma U_\sigma$ satisfies the required properties. \square

3.2 Defining Equation for the Hilbert Scheme

Let \mathcal{Y} be the subspace $\mathcal{Y} \subset \mathbb{C}^{1|1}$ generated by the ideal $I = (x^p, x^q\theta)$. Consider the embedding

$$\begin{array}{ccccc} \mathcal{Y} & \hookrightarrow & \tilde{\mathcal{Y}} & \hookrightarrow & \mathbb{C}^{1|1} \times \mathbb{C}^{p+q|p+q} \\ \downarrow & & \downarrow & & \swarrow \\ \text{Spec } \mathbb{C} & \longrightarrow & \mathbb{C}^{p+q|p+q} & & \end{array}$$

where $\tilde{\mathcal{Y}}$ is the subspace defined by the ideal

$$\tilde{I} = (f := x^p + \sum_{i=0}^{p-1} a_i x^i + \sum_{i=0}^{q-1} \alpha_i x^i \theta, g := x^q \theta + \sum_{i=0}^{q-1} b_i x^i \theta + \sum_{i=0}^{p-1} \beta_i x^i)$$

and $(a_0, \dots, a_{p-1}, b_0, \dots, b_{q-1} | \alpha_0, \dots, \alpha_{q-1}, \beta_0, \dots, \beta_{p-1})$ are coordinates on $\mathbb{C}^{p+q|p+q}$.

Theorem 3.6. $\mathbb{C}_{(p,q)}^{p+q|p+q}$ is isomorphic to $\mathbb{C}^{p|p}$.

Proof. To make a calculation easier, we need to change coordinates. First, apply the long division with the divisor $x^q + \sum_{i=0}^{q-1} b_i x^i$.

$$\begin{aligned} f &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(x^{p-q} + \sum_{i=0}^{p-q-1} c'_i x^i) + \sum_{i=0}^{q-1} d'_i x^i + \sum_{i=0}^{q-1} \gamma_i x^i \theta \\ g &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(\theta + \sum_{i=0}^{p-q-1} \delta_i x^i) + \sum_{i=0}^{q-1} \epsilon_i x^i \end{aligned}$$

Use coordinate change to make this form

$$\begin{aligned} f &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(x^{p-q} + \sum_{i=0}^{p-q-1} a_i x^i) + \sum_{i=0}^{q-1} c_i x^i + \sum_{i=0}^{q-1} \beta_i x^i (\theta + \sum_{i=0}^{p-q-1} \alpha_i x^i) \\ g &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(\theta + \sum_{i=0}^{p-q-1} \alpha_i x^i) + \sum_{i=0}^{q-1} \gamma_i x^i \end{aligned}$$

Let \mathcal{Z} be the restriction of $\tilde{\mathcal{Y}}$ to $\mathbb{C}_{(p,q)}^{p+q|p+q}$.

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & \mathcal{Z} & \longrightarrow & \tilde{\mathcal{Y}} \\ \downarrow & & \downarrow \pi & & \downarrow \\ \text{Spec}(\mathbb{C}) & \longrightarrow & \mathbb{C}_{(p,q)}^{p+q|p+q} & \longrightarrow & \mathbb{C}^{p+q|p+q} \end{array}$$

Let $\phi : \mathcal{O}_U^p \oplus \Pi \mathcal{O}_U^q \rightarrow \pi_* \mathcal{O}_{\mathcal{Z}}|_U$ be the map sending $(\dots, A_i, \dots | \dots, \mathcal{A}_j, \dots)$ to $\sum_{i=0}^{p-1} A_i x^i + \sum_{j=0}^q \mathcal{A}_j x^j \theta$. As in the proof of the theorem 3.5, there is an open set $U \subset \mathbb{C}^{p+q|p+q}$ and the exact sequence

$$\mathcal{O}_U^s \oplus \Pi \mathcal{O}_U^t \xrightarrow{\sigma} \mathcal{O}_U^p \oplus \Pi \mathcal{O}_U^q \xrightarrow{\phi} \pi_* \mathcal{O}_{\mathcal{Z}}|_U \rightarrow 0$$

such that $\mathbb{C}_{(p,q)}^{p+q|p+q}$ is generated by $\sigma = (\sigma_{ij})$.

First of all, compute two elements of $\ker \phi$. For simplicity, denote

$\sum_{i=0}^{p-q-1} a_i x^i, \sum_{i=0}^{q-1} b_i x^i, \dots$ by a, b, \dots .

$$\begin{aligned}
& f(\theta + \alpha) - g(x^{p-q} + a) \\
&= c(\theta + \alpha) - \gamma(x^{p-q} + a) \\
&= \left(\sum_{i=0}^{q-1} c_i x^i \right) \theta + \left(\sum_{i=0}^{q-1} c_i x^i \right) \left(\sum_{i=0}^{p-q-1} \alpha_i x^i \right) - \sum_{i=0}^{q-1} \gamma_i x^i (x^{p-q} + \sum_{i=0}^{p-q-1} a_i x^i) \\
&= g(\theta + \alpha) \\
&= \gamma(\theta + \alpha) \\
&= \left(\sum_{i=0}^{q-1} \gamma_i x^i \right) \theta + \left(\sum_{i=0}^{q-1} \gamma_i x^i \right) \left(\sum_{i=0}^{p-q-1} \alpha_i x^i \right)
\end{aligned}$$

Hence, we find two elements of the kernel

$$h := ((c_0 \alpha_0 - a_0 \gamma_0, \dots, \gamma_{q-1}, \overbrace{0, \dots, 0}^{p-q}), (c_0, \dots, c_{q-1}))$$

and

$$k := ((\gamma_0 \alpha_0, \dots, \gamma_{q-1} \alpha_{p-q-1}, \overbrace{0, \dots, 0}^q), (\gamma_0, \dots, \gamma_{q-1}))$$

Since $\mathbb{C}_{(p,q)}^{p+q|p+q}$ is contained in $\mathcal{H} := Z\left((c_i, \gamma_i)_{i=0}^{q-1}\right) \subset \mathbb{C}^{p+q|p+q}$, we can shrink $\mathbb{C}^{p+q|p+q}$ to \mathcal{H} and repeat the same process.

Then there is another short exact sequence and an open set U

$$\mathcal{O}_U^{s'} \oplus \Pi \mathcal{O}_U^{t'} \xrightarrow{\sigma_{\mathcal{H}}} \mathcal{O}_U^p \oplus \Pi \mathcal{O}_U^q \xrightarrow{\phi_{\mathcal{H}}} \pi_* \mathcal{O}_{\mathcal{Z}} \rightarrow 0$$

Pick an element in the kernel

$$\begin{aligned}
& \sum_{i=0}^{p-1} A_i x^i + \theta \sum_{i=0}^{q-1} B_i x^i \\
&= C f + D g \\
&= C(x^q + b)(x^{p-q} + a) + C \beta \alpha + C \beta \theta + D \theta(x^q + b) + D \alpha(x^q + b)
\end{aligned}$$

where $A_i, B_j \in \Gamma(U, \mathcal{O}_{\mathbb{C}^{p+q|p+q}})$ and $C, D \in \Gamma(\mathbb{C}^{1|1} \times U, \mathcal{O}_{\mathbb{C}^{1|1} \times \mathbb{C}^{p+q|p+q}})$.

Then we get

$$\begin{aligned}
\sum_{i=0}^{p-1} A_i x^i &= C(x^q + \sum_{i=0}^{q-1} b_i x^i)(x^{p-q} + \sum_{i=0}^{p-q-1} a_i x^i) \\
&+ C(\sum_{i=0}^{q-1} \beta_i x^i)(\sum_{i=0}^{p-q-1} \alpha_i x^i) + D(\sum_{i=0}^{p-q-1} \alpha_i x^i)(x^q + \sum_{i=0}^{q-1} b_i x^i)
\end{aligned} \tag{1}$$

$$\sum_{i=0}^{q-1} B_i x^i = C \left(\sum_{i=0}^{q-1} \beta_i x^i \right) + D(x^q + \sum_{i=0}^{q-1} b_i x^i) \quad (2)$$

By comparing coefficient of x^p in 1, we can see $C = 0$. Similarly, from 2 we get $D = 0$. Therefore, $A_i = B_i = 0$ for all i .

Therefore, ϕ is an isomorphism and $\mathbb{C}_{(p,q)}^{p+q|p+q} = \mathcal{H}$ is defined by the ideal (σ_{ij}) where

$$\sigma = \left(\begin{array}{ccc|ccc} c_0 \alpha_0 - a_0 \gamma_0 & \cdots & 0 & c_0 & \cdots & c_{q-1} \\ & \gamma_0 \alpha_0 & \cdots & \gamma_0 & \cdots & \gamma_{q-1} \end{array} \right)$$

I.e., $(\sigma_{ij}) = (c_0, \dots, c_{q-1}, \gamma_0, \dots, \gamma_{q-1})$.

Moreover, $\mathbb{C}_{(p,q)}^{p+q|p+q} \simeq \mathbb{C}^{p|p}$. □

Theorem 3.7. $\mathbb{C}^{p|p}$ represents the Hilbert functor $\mathcal{H}_{\mathbb{C}^{1|1}}^{p|q}$.

Proof. Pick any flat family in $\mathcal{H}_{\mathbb{C}^{1|1}}^{p|q}(B)$.

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad} & \mathbb{C}^{1|1} \times B \\ & \searrow p & \downarrow \\ & & B \end{array}$$

By the lemma 3.1, \mathcal{Y} is defined by an ideal

$$\left(x^p + \sum_{i=0}^{p-1} c_i x^i + \sum_{i=0}^{q-1} \gamma_i x^i \theta, x^q \theta + \sum_{i=0}^{q-1} d_i x^i \theta + \sum_{i=0}^{p-1} \delta_i x^i \right)$$

where $c_i, d_i \in (H^0(B, \mathcal{O}_B))^0, \gamma_i, \delta_i \in (H^0(B, \mathcal{O}_B))^1$. Then there is a natural map $B \rightarrow \mathbb{C}^{p+q|p+q}$ and this map factors through $\mathbb{C}_{(p,q)}^{p+q|p+q}$ since the map p is flat. Observe that p is the pull back of π and such a map is unique. □

From now on, we will fix coordinate

$$(a_0, \dots, a_{p-q-1}, b_0, \dots, b_{q-1} \mid \alpha_0, \dots, \alpha_{p-q-1}, \beta_0, \dots, \beta_{q-1})$$

on $\text{Hilb}^{p|q}(\mathbb{C}^{1|1}) \simeq \mathbb{C}^{p|p}$ as

$$\begin{aligned} \left(\begin{aligned} f &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(x^{p-q} + \sum_{i=0}^{p-q-1} a_i x^i) + \sum_{i=0}^{q-1} \beta_i x^i (\theta + \sum_{i=0}^{p-q-1} \alpha_i x^i) \\ g &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(\theta + \sum_{i=0}^{p-q-1} \alpha_i x^i) \end{aligned} \right) \end{aligned}$$

4 Families of 0-dimensional subspaces on supercurves

Let S be a smooth supercurve. By applying the theorem 3.7 to a suitable representable open cover of $\mathcal{H}_S^{p|q}$, we can show the representability of the Hilbert functor $\mathcal{H}_S^{p|q}$. Note that $\left(\mathrm{Hilb}^{p|q}(S)\right)_{red} = \mathrm{Hilb}^p(S_{red})$ and hence the finiteness and Hausdorff conditions hold automatically.

Proof of the Theorem 2.5.

Proof. Let $U = \{U_i\}_{i=1}^r$ be a set of r disjoint open subsets of S such that each U_i is isomorphic to some nonempty open subset of \mathbb{C}^1 . For such U , we can define an open subfunctor

$$\mathcal{H}_{S,U}^{p|q} := \coprod_{\substack{\sum p_i = p \\ \sum q_i = q}} \bigcup_i \mathcal{H}_{U_i}^{p_i|q_i}$$

Observe the following facts.

Fact 1: $\mathcal{H}_S^{p|q} = \bigcup_U \mathcal{H}_{S,U}^{p|q}$.

Fact 2: Each $\mathcal{H}_{S,U}^{p|q}$ is an open subfunctor of $\mathcal{H}_S^{p|q}$ and representable by the smooth superspace of dimension $(p|p)$.

Therefore, the Hilbert functor $\mathcal{H}_S^{p|q}$ is representable by a dimension $(p|p)$ smooth superspace.

□

For the ordinary Hilbert scheme of points, the Hilbert scheme $\mathrm{Hilb}^4(\mathbb{C}^3)$ is not smooth. We can check this by check the non-smoothness of $\mathrm{Hilb}^4(\mathbb{C}^3)$ at $I = m^2 = (x, y, z)^2$. In my PhD thesis, I'll deal with smoothness or non-smoothness of the Hilbert scheme $\mathrm{Hilb}^{p|q}(\mathbb{C}^{1|2})$. Actually, it turns out that $\mathrm{Hilb}^{p|q}(\mathbb{C}^{1|2})$ is not smooth for certain cases.

5 Non-splitness of the Hilbert scheme

In the previous sections, we not only show the existence of the Hilbert schemes but also find the local structure. As an application, we check the splitness of the Hilbert scheme.

Example 5.1. Consider a line bundle $V = \mathcal{O}_{\mathbb{P}^1}(k)$ on \mathbb{P}^1 . The supermanifold ΠV is a smooth supercurve.

Observe that $\left(\text{Hilb}^{1|1}(\Pi V)\right)_b = \mathbb{P}^1$ has standard affine open cover $\mathbb{P}^1 = U_0 \cup U_1$ and we can assign affine coordinates on each U_i

$$\Pi V|_{U_0} \simeq \mathbb{C}_{x,\theta}^{1|1}$$

$$\Pi V|_{U_1} \simeq \mathbb{C}_{y,\psi}^{1|1}$$

Then we have $\text{Hilb}^{1|1}(\Pi V)|_{U_0} \simeq \mathbb{C}_{a,\alpha}^{1|1}$ and $\text{Hilb}^{1|1}(\Pi V)|_{U_1} \simeq \mathbb{C}_{b,\beta}^{1|1}$, from the Theorem 3.7. From the already known relations $x = a + \alpha\theta$, $y = b + \beta\psi$, $y = 1/x$, $\psi = \theta/x^k$ and $b = \frac{1}{a}$ on the intersection $U_0 \cap U_1$, we can compute the transition map $\beta = -a^{k-2}\alpha$. Therefore, $\text{Hilb}^{1|1}(\Pi V) = \Pi W$ where $W = \mathcal{O}(-k+2) = \mathcal{O}(2) \otimes V^\vee$ and $\text{Hilb}^{1|1}(\Pi V)$ is split.

Let $V = \mathcal{O}_{\mathbb{P}^1}(k)$ be a line bundle on \mathbb{P}^1 . We will show non-splitness of the Hilbert scheme $\text{Hilb}^{2|1}(\Pi V)$. Note that the bosonic part of $\text{Hilb}^{2|1}(\Pi V)$ is $\mathbb{P}^1 \times \mathbb{P}^1$. We can see this simply by modding out by the odd part.

Let $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal. Let $U_{ij} = U_i \times U_j \subset \mathbb{P}^1_{[z_0, z_1]} \times \mathbb{P}^1_{[w_0, w_1]}$ be an open subset, where U_i is defined by $z_i \neq 0$ and U_j is defined by $w_j \neq 0$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ has another open cover

$$\mathbb{P}^1 \times \mathbb{P}^1 = U_{00} \cup (U_{10} - \Delta) \cup (U_{01} - \Delta) \cup U_{11}$$

Define $V_1 := U_{00}$, $V_2 := U_{10} - \Delta$, $V_3 := U_{01} - \Delta$ and $V_4 := U_{11}$.

Define p_{10} and p_{01} to be the projections to the reduced parts

$$p_{10} : \text{Hilb}^{1|1}(\Pi V|_{U_1}) \times \text{Hilb}^{1|0}(\Pi V|_{U_0}) \rightarrow U_1 \times U_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$$

$$p_{01} : \text{Hilb}^{1|1}(\Pi V|_{U_0}) \times \text{Hilb}^{1|0}(\Pi V|_{U_1}) \rightarrow U_0 \times U_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$$

Then we can define a pullback $\Delta^* := p^* \Delta$, for each $p = p_{10}, p_{01}$.

First, observe that there are natural inclusion maps

$$\text{Hilb}^{1|1}(\Pi V|_{U_1}) \times \text{Hilb}^{1|0}(\Pi V|_{U_0}) - \Delta^* \xrightarrow{\sim} \text{Hilb}^{2|1}(\Pi V)|_{V_2} \hookrightarrow \text{Hilb}^{2|1}(\Pi V)$$

$$\text{Hilb}^{1|1}(\Pi V|_{U_0}) \times \text{Hilb}^{1|0}(\Pi V|_{U_1}) - \Delta^* \xrightarrow{\sim} \text{Hilb}^{2|1}(\Pi V)|_{V_3} \hookrightarrow \text{Hilb}^{2|1}(\Pi V)$$

From the above inclusions, we can easily see that the Hilbert scheme $\text{Hilb}^{2|1}(\Pi V)$ can be covered by four open subspaces

$$\text{Hilb}^{2|1}(\Pi V)|_{V_1} \cup \text{Hilb}^{2|1}(\Pi V)|_{V_2} \cup \text{Hilb}^{2|1}(\Pi V)|_{V_3} \cup \text{Hilb}^{2|1}(\Pi V)|_{V_4}$$

To make this argument complete, we need to glue all open subsets. Let us start with gluing V_1 and V_3 . On each open set U_i , we can trivialize and assign coordinates of ΠV .

$$\begin{aligned}\Pi V|_{U_0} &\simeq \mathbb{C}_{x,\theta}^{1|1} \\ \Pi V|_{U_1} &\simeq \mathbb{C}_{y,\psi}^{1|1}\end{aligned}$$

Assign coordinates induced from the Section 3.2

$$\begin{aligned}\mathrm{Hilb}^{2|1}(\Pi V)\Big|_{V_3} &\simeq \mathrm{Hilb}^{1|1}(\Pi V|_{U_0}) \times \mathrm{Hilb}^{1|0}(\Pi V|_{U_1}) - \Delta^* \\ &\simeq \mathbb{C}_{c_1|\gamma_1}^{1|1} \times \mathbb{C}_{c_2|\gamma_2}^{1|1} - \tilde{\Delta} \\ \mathrm{Hilb}^{2|1}(\Pi V)\Big|_{V_1} &\simeq \mathbb{C}_{a_1,a_2|\alpha_1,\alpha_2}^{2|2}\end{aligned}$$

where $\tilde{\Delta}$ is defined by $c_1 c_2 = 1$.

On the intersection $V_1 \cap V_3$, we have $c_2 \neq 0$ and identities $y = \frac{1}{x}$ and $\psi = \frac{\theta}{x^k}$. Compute the gluing map $\mathbb{C}_{c_1|\gamma_1}^{1|1} \times \mathbb{C}_{c_2|\gamma_2}^{1|1} - \Delta \rightarrow \mathbb{C}_{a_1,a_2|\alpha_1,\alpha_2}^{2|2}$ to be the isomorphism induced by the following calculation

$$\begin{aligned}((c_1|\gamma_1), (c_2|\gamma_2)) &\mapsto \langle x + c_1 + \gamma_1 \theta \rangle \times \langle y + c_2, \psi + \gamma_2 \rangle \\ &\mapsto \langle (x + c_1 + \gamma_1 \theta)(y + c_2), (x + c_1 + \gamma_1 \theta)(\psi + \gamma_2) \rangle \\ &= \left\langle (x + c_1 + \gamma_1 \theta)\left(x + \frac{1}{c_2}\right), (x + c_1 + \gamma_1 \theta)\left(\theta + \frac{\gamma_2}{(-c_2)^k}\right) \right\rangle \\ &= \left\langle \left(x + c_1 - \gamma_1 \gamma_2 (-c_2)^{-k}\right)(x + c_2^{-1}) + \gamma_1 (c_2^{-1} - c_1)(\theta + \gamma_2 (-c_2)^{-k}), \right. \\ &\quad \left. \left(x + c_1 - \gamma_1 \gamma_2 (-c_2)^{-k}\right)(\theta + \gamma_2 (-c_2)^{-k}) \right\rangle \\ &\mapsto \left(c_1 - \gamma_1 \gamma_2 (-c_2)^{-k}, \frac{1}{c_2} \middle| \gamma_1 \left(\frac{1}{c_2} - c_1 \right), \gamma_2 (-c_2)^{-k} \right)\end{aligned}\tag{3}$$

One can similarly compute gluing maps on each intersection $V_i \cap V_j$ for all i and j , and easily check the transitivity.

Let W be the vector bundle on $(\mathrm{Hilb}^{2|1}\Pi V)_b$ defined by $W^\vee = \mathcal{I}/\mathcal{I}^2$, where \mathcal{I} is the ideal sheaf of $\mathrm{Hilb}^{2|1}\Pi V$ generated by nilpotents. To check the non-splitness of the $\mathrm{Hilb}^{2|1}\Pi V$, it is enough to find the obstruction class $\omega_2 =$

$w(\varphi^{(1)}) \in H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee)$ and check that it is not vanishing. ([4])

Since $\wedge^2 W^\vee$ is a line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$, there are some a and b such that $\wedge^2 W^\vee \simeq \mathcal{O}(a, b)$.

From the computation (3), we know that the transition map on $V_1 \cap V_3$ is

$$\begin{aligned} a_1 &\mapsto c_1 - \gamma_1 \gamma_2 (-c_2)^{-k} \\ a_2 &\mapsto \frac{1}{c_2} \\ \alpha_1 &\mapsto \gamma_1 \left(\frac{1}{c_2} - c_1 \right) \\ \alpha_2 &\mapsto \gamma_2 (-c_2)^{-k} \end{aligned} \tag{4}$$

Assign coordinates,

$$\text{Hilb}^{2|1}(\Pi V)|_{V_2} \simeq \mathbb{C}_{b_1|\beta_1}^{1|1} \times \mathbb{C}_{b_2|\beta_2}^{1|1} - \Delta^*$$

For b and β 's, we have equations

$$x + b_2 = 0, \theta + \beta_2 = 0 \text{ and } y + b_1 + \beta_1 \psi = 0$$

On $V_1 \cap V_2$, by using the identities $xy = 1$ and $\psi = \theta/x^k$, we get

$$\langle y + b_1 + \beta_1 \psi \rangle = \left\langle x + b_1^{-1} - \beta_1 (-b_1)^{k-2} \theta \right\rangle$$

Then we can compute that

$$\left\langle \left(x + b_1^{-1} - \beta_1 (-b_1)^{k-2} \theta \right), (x + b_2, \theta + \beta_2) \right\rangle$$

corresponds to the ideal

$$\begin{aligned} \left\langle \left(x + \frac{1}{b_1} + \beta_1 (-b_1)^{k-2} \beta_2 \right) (x + b_2) - \beta_1 (-b_1)^{k-2} \left(b_2 - \frac{1}{b_1} \right) (\theta + \beta_2), \right. \\ \left. \left(x + \frac{1}{b_1} + \beta_1 (-b_1)^{k-2} \beta_2 \right) (\theta + \beta_2) \right\rangle \end{aligned}$$

By comparing above ideal with

$$\langle (x + a_1)(x + a_2) + \alpha_1(\theta + \alpha_2), (x + a_1)(\theta + \alpha_2) \rangle$$

we can check that the transition map is

$$\begin{aligned}
a_1 &\mapsto \frac{1}{b_1} + \beta_1 \beta_2 (-b_1)^{k-2} \\
a_2 &\mapsto b_2 \\
\alpha_1 &\mapsto -\beta_1 (-b_1)^{k-2} (b_2 - \frac{1}{b_1}) \\
\alpha_2 &\mapsto \beta_2
\end{aligned} \tag{5}$$

Transition maps for $V_{24} := V_2 \cap V_4$ and $V_{34} := V_3 \cap V_4$ can be computed from transition maps for V_{13} and V_{12} by changing variables.

For a and b , where $\wedge^2 W^\vee \simeq \mathcal{O}(a, b)$, we have the following lemma.

Lemma 5.1. $a = k - 3$ and $b = -k - 1$

Proof. Restrict $\wedge^2 W^\vee$ to $\mathbb{P}^1 \times \{0\}$. Then the transition map on $V_1 \cap V_2$ gives the transition map on $\mathbb{P}^1 \times \{0\} \simeq \mathbb{P}^1$. Change coordinates on V_2 by $\beta_1(b_1 b_2 - 1) \mapsto \beta_1$, then the transition map (5) gives us $\alpha_1 \alpha_2 = \beta_1 \beta_2 (-b_1)^{k-3}$ and $a = k - 3$. To find b , we need to restrict the line bundle to $\{0\} \times \mathbb{P}^1$. Then a transition map on $V_2 \cap V_4$ gives

$$\begin{aligned}
\delta_1 &\mapsto \frac{\beta_1}{b_2} \\
\delta_2 &\mapsto -(-b_2)^{-k} \beta_2
\end{aligned}$$

Note that, by setting $b_1 = 0$, this transition map can be derived from the transition map of $\text{Hilb}^{2|1}(\text{IV})$ on $V_2 \cap V_4$. From the transition map, we get $\delta_1 \delta_2 = -\beta_1 \beta_2 (-b_2)^{-k-1}$ and $b = -k - 1$. □

Theorem 5.2. *Let V be the line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$ on \mathbb{P}^1 . For any k , the Hilbert scheme $\text{Hilb}^{2|1} \text{IV}$ is non-split.*

Proof.

It is enough to show that the obstruction class $\Psi \in H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{T} \otimes \wedge^2 W^\vee)$ defined by $\text{Hilb}^{2|1} \text{IV}$ is non-zero.

1. On $V_{12} := V_1 \cap V_2$, the transition map (5) defines

$$\Psi_{12} = -\frac{\alpha_1 \alpha_2}{a_2 - a_1} \frac{\partial}{\partial a_1}$$

2. On $V_{13} := V_1 \cap V_3$, the transition map (4) gives

$$\Psi_{13} = -(-c_2)^{-k} \gamma_1 \gamma_2 \frac{\partial}{\partial a_1} = -\frac{\alpha_1 \alpha_2}{a_2 - a_1} \frac{\partial}{\partial a_1}$$

3. On $V_{23} := V_2 \cap V_3$, we have $\Psi_{23} = 0$ because $V_{23} \subset V_{12} \cap V_{13}$.

Now, we need to show that Ψ is non-zero.

Suppose that there are σ_i 's such that $\Psi_{ij} = \sigma_j - \sigma_i$ on each V_{ij} . Then we can find $f(\frac{z_1}{z_0}, \frac{w_1}{w_0}) \in k\left[\frac{z_1}{z_0}, \frac{w_1}{w_0}\right]$, $g(\frac{z_0}{z_1}, \frac{w_1}{w_0}) \in k\left[\frac{z_0}{z_1}, \frac{w_1}{w_0}\right]$ and $h(\frac{z_1}{z_0}, \frac{w_0}{w_1}) \in k\left[\frac{z_1}{z_0}, \frac{w_0}{w_1}\right]$ such that

$$\begin{aligned} \sigma_1 &= f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \alpha_1 \alpha_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + f'\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \alpha_1 \alpha_2 \frac{\partial}{\partial(\frac{w_1}{w_0})} \\ \sigma_2 &= g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) \beta_1 \beta_2 \frac{\partial}{\partial(\frac{z_0}{z_1})} + g'\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) \beta_1 \beta_2 \frac{\partial}{\partial(\frac{w_1}{w_0})} \\ \sigma_3 &= h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) \gamma_1 \gamma_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + h'\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) \gamma_1 \gamma_2 \frac{\partial}{\partial(\frac{w_0}{w_1})} \end{aligned}$$

Observe that

$$\begin{aligned} \Psi_{12} &= -\left(-\frac{z_0}{z_1}\right)^{k-2} \beta_1 \beta_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} \\ &= -f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \left(b_2 - \frac{1}{b_1}\right) (-b_1)^{k-2} \beta_1 \beta_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} \\ &\quad + g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) \left(\frac{z_1}{z_0}\right)^2 \beta_1 \beta_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + (\dots) \frac{\partial}{\partial(\frac{w_1}{w_0})} \end{aligned}$$

Therefore, we have

$$-\left(-\frac{z_0}{z_1}\right)^k = -g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) + f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right) \left(-\frac{z_0}{z_1}\right)^k \quad (6)$$

Similarly, Ψ_{13} gives

$$-\left(-\frac{w_1}{w_0}\right)^k = h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) - f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right) \left(-\frac{w_1}{w_0}\right)^k \quad (7)$$

Also, Ψ_{23} gives us

$$h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) - g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) \left(-\frac{w_1}{w_0}\right)^k \left(-\frac{z_1}{z_0}\right)^k = 0 \quad (8)$$

Finally, we will derive a contradiction for any k .

- I . If k is positive, $g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) \cdot \left(-\frac{w_1}{w_0}\right)^k \left(-\frac{z_1}{z_0}\right)^k$ have a term with w_0 at the denominator for $g \neq 0$. To make the equation (8) true, g and h must be zero. However, the equation (7) implies that

$$f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \cdot \left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right) = -1$$

which is a contradiction.

- II . If $k < 0$, $g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) \cdot \left(-\frac{w_1}{w_0}\right)^k \left(-\frac{z_1}{z_0}\right)^k$ has z_1 at the denominator for $g \neq 0$. In a similar way to the case $k > 0$, we can derive a contradiction.

- III . If $k = 0$, the equation (8) becomes $h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) = g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right)$. Therefore, $h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) = g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) = c$ for some constant c . Then

$$(6) \Rightarrow f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \cdot \left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right) - c = -1$$

The only possible case is $f = 0$ and $c = 1$. Plug in $f = 0$ and $h = 1$ to (7) and then we get a contradiction.

Hence, we show that the obstruction class Ψ is nonzero.

□

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